Proof that A383250 is the Sorted Version of A342045

Theorem

Theorem 1. *A383250 is the sorted version of A342045. That is,*

$$\boxed{A383250 = \textit{sorted}(A342045)}.$$

Proof

We proceed by precise definitions and logical steps.

1. Definitions

Definition 1 (Set $\mathcal{A}$). *Define*

$$\mathcal{A} = \{n \in \mathbb{N}_0 \mid \text{(i) } \forall i,\ d_i \text{ odd } \Rightarrow d_{i+1} < d_i, \text{ and (ii) } d_k \neq 1\},$$

where $d_1, d_2, \ldots, d_k$ are the decimal digits of n from left to right.

Definition 2 (Extendable Set). *A finite set $F \subseteq \mathbb{N}_0$ is* extendable *if*

$$\forall \text{ finite } F' \subseteq F,\ \exists m \in \mathbb{N}_0 \backslash F' \text{ such that } F' \cup \{m\} \text{ satisfies conditions (i) and (ii).}$$

Definition 3 (Greedy Construction of A342045). *The sequence A342045 is generated by:*

- *Starting with $a_1 = 0$.*
- *At each step $n \geq 1$, choosing the smallest nonnegative integer k not yet selected such that adding k preserves extendibility.*

2. Lemma 1: Greedy Only Selects Numbers in $\mathcal{A}$

Lemma 1. *Every number selected by the greedy construction A342045 belongs to $\mathcal{A}$.*

Proof. Suppose $k \in \mathbb{N}_0$ is considered for selection.

- If k violates condition (i), it cannot be selected: the violation is local and cannot be repaired by any extension.
- If k ends with digit 1, then k is disallowed: no future extensions involving k would satisfy the rules.

Thus, only numbers satisfying conditions (i) and (ii) can be selected.
Hence,

$$\mathrm{set}(\mathrm{A342045}) \subseteq \mathcal{A}.$$

□

3. Lemma 2: Every Number in $\mathcal{A}$ Appears in A342045

Lemma 2 (Monotonicity of extendibility)**.** *If F is an extendable finite set and $n \in \mathcal{A}$, then $F \cup \{n\}$ is extendable.*

Proof. Let $F' \subseteq F \cup \{n\}$ be any finite subset.

- If $n \notin F'$, then $F' \subseteq F$, and extendibility of F provides the required m.
- If $n \in F'$, then $F' \setminus \{n\} \subseteq F$ is finite, and again extendibility of F provides the required m.

Since $n \in \mathcal{A}$, and $\mathcal{A}$ is closed under the rules, the extension is valid.
Thus,

$$F \text{ extendable } \Rightarrow F \cup \{n\} \text{ extendable.}$$

□

Lemma 3 (Eventual Selection)**.** *Every $n \in \mathcal{A}$ eventually appears in A342045.*

Proof. Proceed by induction on $n \in \mathbb{N}_0$.
Base case: $n = 0$ is selected first.
Inductive step: Assume all $m < n$ with $m \in \mathcal{A}$ have been selected.
When considering n, all smaller numbers have been chosen.
By monotonicity of extendibility (Lemma 2), adding n to the current list maintains extendibility.
Thus, n is eligible and selected at that time.
Hence,

$$\mathcal{A} \subseteq \mathrm{set}(\mathrm{A342045}).$$

□

4. Set Equality and Sorting

Combining Lemmas 1 and 3, we obtain:

$$\mathrm{set}(\mathrm{A342045}) = \mathcal{A}.$$

Since A342045 lists elements of $\mathcal{A}$ in greedy order, and A383250 lists $\mathcal{A}$ in increasing order, it follows that:

$$\boxed{\mathrm{A383250} = \mathrm{sorted}(\mathrm{A342045})}.$$

References

[1] OEIS Foundation Inc. (2025), *The On-Line Encyclopedia of Integer Sequences*, published electronically at `https://oeis.org`.
A342045: `https://oeis.org/A342045`
A383250: `https://oeis.org/A383250`